Multiplicative Thinking Tools

Common Misunderstanding

Although most students at this level have some knowledge of the multiplication facts to 100 and can perform simple multiplication and division procedures correctly, many rely on rote learning and/or a naïve, *groups of* understanding for multiplication based on repeated addition (often counting equal groups by ones). With little or no access to a broader range of ideas for multiplication they find it difficult to develop efficient mental strategies, and as a consequence, tend to rely on memorised procedures for multiplying and dividing larger whole numbers and decimals.

This could be due to/associated with:

- an inability to trust the count and see numbers as countable units in their own right, that is, view 6 items as 1 six ("a six") rather than 6 ones (see Tool 2.2);
- poorly developed or non-existent mental strategies for addition and subtraction;
- an over-reliance on physical models to solve simple multiplication problems; and/or
- a limited exposure to alternative models of multiplication.

Students need to be able to think about multiplication in a number of different ways to recognise when multiplication is required and how it relates to division, support efficient mental and written computation, and solve a wider range of problems involving equal groups, simple proportion, combinations, and rate. To do this they need to recognise the numbers 2 to 10 as countable units, count large collections more efficiently, and appreciate the advantages of representing multiplicative situations in terms of arrays and regions. That is, that arrays and regions

- more neutrally represent all aspects of the multiplicative situation, that is, the number of groups, the equal number in each group, and the product (last two not as evident in *groups of* models);
- can be used to relate the two ideas for division, *partition* (or sharing) and *quotition* (or how many groups in), to multiplication;
- support commutativity (e.g. 3 fours can be rotated to show that it is the same as 4 threes) so halving the amount of learning required for the multiplication facts;
- support more efficient, generalisable mental strategies for multiplication; and
- provide a basis for moving from a count of equal groups (e.g. 1 six, 2 sixes, 3 sixes, 4 sixes, …) to a constant number of groups (e.g. 6 ones, 6 twos, 6 threes, 6 fours, 6 fives …) which supports more efficient mental strategies (e.g. 6 groups of anything is double 3 groups or 5 groups and 1 more group).

More importantly, arrays and regions support the shift from an additive *groups of* model to a *factor-factor-product* model which is needed to support fraction representation, the multiplication and division of larger whole numbers, fractions and decimals, and algebra. An awareness of the "*for each*" idea or Cartesian product is also needed at this level to support work in Chance and Data (e.g., problems involving combinations), measurement

(including problems involving rates), and fraction representation. For example, if a diagram showing thirds is halved and halved again, there are 4 smaller parts for each third; this is not a groups of idea that corresponds to students' experience.

A key indicator of the extent to which students have developed a broader range of ideas to support multiplicative thinking is the extent to which they manipulate both the size of the group and the number of groups to meet specific needs (e.g. instead of committing 6 eights to memory in a meaningless or rote way, recognise that this can be thought of as 5 eights and 1 more eight, or 3 eights doubled).

3.1 Countable Units Tool[1]

MATERIALS

- 12 square tiles glued together in pairs one on top of the other and displayed on a card so that one pair can be exposed without exposing the others (e.g. see below).

- Something to hide the tiles, e.g. a paper plate or piece of card.
- A single square tile.
- 87 Unifix blocks in a container.

INSTRUCTIONS

Bold type indicates what should be said.

Say: **"I'm interested in how people count by threes. If you count to twelve by threes, how many threes would you count?** Observe student's response, in particular, the extent to which he/she marks the count in some way (e.g. by using fingers). If the student clearly needs to model the count in some way, ask him/her to explain their strategy then proceed to the square tiles task. If completed quickly and easily, omit the square tiles task and proceed to the Unifix task.

Ask the student to close his/her eyes while the 12 tiles, glued together in pairs, are placed under the paper plate. Ask the student to open his/her eyes and while showing the student the single tile.

Say: **"There are twelve of these tiles under the paper plate"**. Carefully slide the paper plate to show the student a single pair of tiles replace and ask:

"How many twos do you think are under here?" Observe student's response, in particular, the extent to which he/she marks the count. If done relatively easily, proceed to the Unifix task.

Place 87 Unifix blocks in front of the student.

Say: **"There are 87 blocks in this container. Can you make sticks of ten?"** If student moves to make sticks of ten, interrupt after one stick has been made and ask:

"How many sticks of ten could you make like that?" Note student's response. If the student suggests or continues making sticks of ten, interrupt again.

Say: **"Let's just pretend that you have made all the sticks, how many would there be?"** Observe student's response.

[1] Adapted from Killion, K., Steffe, L. & Stanic, G. (1989). Children's multiplication. *Arithmetic Teacher*, September, 34-36.

3.2 Additive Strategies Tool

MATERIALS

- 5 Cards (see Multiplicative Thinking Resources).

INSTRUCTIONS

Show Card 1.

Say: "**This is a different way of showing what we get when we add up these numbers."** Indicate 3, 2 and 4, then point to 9.

Say: **"Do you agree that if we added these numbers up this will be the answer?"** Indicate 9. If the student agrees and appears to understand the task, proceed to Card 2.

Place Card 2 in front of the student.

Ask: **"The answer is missing from this card. Can you add up the numbers to find the answer please? How did you work that out?"** Note student's response then remove the card. If answered relatively easily proceed to Card 3, otherwise stop at this point.

Place Card 3 in front of the student.

Say: **"This time the answer is there** (point to 24), **but one of the numbers is missing. Can you work out what number is missing please? How did you work that out?"**

If this was done relatively easily, remove Card 3 and place Card 4 in front of the student.

Ask: **"What do you think needs to be done here? Can you do that for me please? Can you tell me how you did that?"** If student hesitates or find this difficult, stop and try to find out why.

If this was done easily, proceed to Card 5 and repeat the questions.

3.3 Sharing Tool

MATERIALS

- 24 counters.
- 6 plastic lolly bags (available from supermarkets) or equivalent.
- 2 Cards (see Multiplicative Thinking Resources).

INSTRUCTIONS

Place 24 counters and 6 lolly-bags in front of student, point to counters.

Say: **"Can you share these out equally so that every lolly-bag has the same amount of counters?"**

Note student's response/strategy.

Place Card A and Card B in front of the student.

Say: **"Without counting can you tell me if there are the same number of dots on each card, or does one card have more than the other? How do you know?"**

If completed relatively easily…

Say: **"Imagine you have 2 lollies but your sister has 3 times as many lollies as you. How many lollies does your sister have?"**

3.4 Array and Regions Tool[2]

MATERIALS

- 2 Cards (see Multiplicative Thinking Resources).
- An A3 sheet of paper and 3 blank "*name-tags*" (or business cards) measuring 9.5 cm x 5.5 cm (this needs to be accurate).

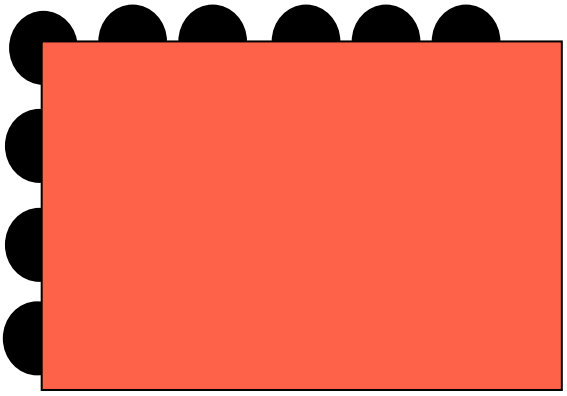
INSTRUCTIONS

Put Card 2 on top of Card 1 so that it is completely covered, place in front of the student.

Say: **"I'm going to show you what is underneath this card very quickly and I want you to tell me what you notice?"** Remove Card 2 and replace as quickly as possible. Note student's response (e.g. "*there are dots in rows*").

Say: "**I'm going to show you some of the dots and hide the others"**. As you say this, slide Card 2 over Card 1 so that only part of the array is shown as in the diagram below.

Ask: **"Can you tell me how many dots there are altogether?"** Note student's response.



If correct (24 dots), remove cards and place the A3 sheet of paper in front of the student together with the 3 blank "*name-tags*".

Say: **"Can you tell me how many name-tags the same size as these could be made from this sheet of paper please?"** Note student's strategy and response. Stop if student appears unsure about how to proceed (e.g. guesses or shows no signs of manipulating cards to determine how many might fit).

[2] A3 task adapted from Simon, M. & Blume, G. (1992). Understanding multiplicative structures: A study of prospective elementary teachers. In W. Geeslin & K. Graham (Eds.) *Proceedings of the Sixteenth PME Conference*, Vol III, pp.3-11. University of New Hampshire: PME

3.5 Cartesian Product Tool[3]

MATERIALS

- Cards (see Multiplicative Thinking Resources).
- Pen and paper.

INSTRUCTIONS

Place card showing T-shirt sign in front of student.

Say: **"This sign says there are 3 different colours and 4 different sizes of T-shirt on special. If you bought one of each type, how many different T-shirts could you buy?"** Note student's response.

If this is done relatively easily, place the lunch order card in front of student.

Say: **"The school canteen offers 2 choices of rolls, 4 choices of filling and 3 choices of drink. Claire ordered a peanut-butter roll and a drink. What might she have ordered?"** Note student's response.

If this done relatively easily…

Say: **"There are 23 students in Claire's class. If everyone ordered a roll with a filling and a drink, could they all have a different lunch order?"** Note student's response. Stop if student appears uncertain; try to find out what is making this difficult.

[3] Canteen problem sourced from the *Scaffolding Numeracy in the Middle Years Project (2003-2006)*

3.6 Proportional Reasoning Tool[4]

MATERIALS

- Card showing a red, blue, and orange eel (see Multiplicative Thinking Resources).

INSTRUCTIONS

Place card showing eels in front of student.

Say: **"This card shows three eels. The blue eel is twice as long as the red eel and the orange eel is three times as long as the red eel. The eels are fed food pellets according to their length. If the red eel gets 2 food pellets, how many pellets would be fed to the other two eels?"** Note the student's response.

If done fairly easily, point to the blue eel and ask:

"If the blue eel gets 14 food pellets, how many pellets would be fed to the other two eels?" Note the student's response.

If done fairly easily, ask:

"If the orange eel gets 18 food pellets, how many pellets would be fed to the other two eels?" Note the student's response.

[4] Adapted from Hart, K. et al.(1981). *Children's understanding of mathematics: 11-16*, (pp.88-101). London: John Murray

Multiplicative Thinking Advice

3.1 Countable Units Tool

This task examines the extent to which students recognise numbers as *abstract composite wholes*[5] (i.e. as countable units in the absence of physical materials/models). The idea that a count can be counted is a difficult notion for some students but it is an essential underpinning for place-value, multiplication and division. This task should only be used where students have demonstrated some capacity to physically count collections by twos, fives and tens.

Observed Response	Interpretation/Suggested Teaching Response
Little/no response to initial imaginary task, some attempt to solve square tile task.	*May not understand task or unable to work with 2 and/or 3 as mental objects.* • Consolidate part-part-whole knowledge using Subitising Cards (see trusting the Count Advice). • Practice counting larger collections by 2s and 5s.
Able to keep track of count of threes and/or twos, but unable to say how many tens would be made.	*Suggests some capacity to work with 2 and/or 3 as mental objects, may not be able to sustain this for 6 or more groups, may not trust the count of 10 or see 2 digit numbers in terms of a count of tens and ones.* • Introduce arrays as a more efficient way to count larger collections, encourage the use of doubling. • Practice *making, naming* and *recording* tens and ones, emphasising the count of tens in the tens place and the count of ones in the ones place (see Booker et al.[6] for further details). • Reinforce notion of ten as a countable unit by counting bundles of ten sticks or MAB tens to 10 tens and beyond (e.g. 1 ten, 2 tens, … 9 tens, 10 tens, 11 tens, 12 tens …23 tens … etc.). Explore patterns in recording, e.g. 6 tens recorded as 60 so 23 tens recorded as 230 (this should NOT be interpreted as "*adding a zero*", but as how we name tens, 0 is recorded to indicate that there are no ones).

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[5] Killion, K., Steffe, L. & Stanic, G. (1989). Children's multiplication. *Arithmetic Teacher*, September, 34-36.
[6] Booker, G., Bond, D., Sparrow, P. & Swan, L. (2003). *Teachng Primary Mathematics*. Melbourne: Pearson-Prentice Hall.

3.1 Countable Units Tool

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Observed Response	Interpretation/Suggested Teaching Response
Able to say how many tens would be made.	*Can work with 2 and 3 as mental objects, appears to understand the basis on which 2-digit numbers are recorded.* • Consolidate 2-digit place-value by *comparing* two numbers presented in different ways (e.g. 3 MAB tens and 17 ones and a card showing 46, which is bigger/smaller? Why/how do you know?), *ordering/sequencing* (order 5 or more 2-digit numbers or place in sequence on a rope from 0 to 100, discuss and refine strategies), *counting forwards and backwards in place-value parts* starting anywhere (e.g. 27, 37, 47 (clap), 46, 45, 44, 43, …), and by *renaming* (e.g. 45 is 4 tens and 5 ones or 45 ones, which is easier to see?). • Practice reading to the tens place to identify the count of tens, consider using open number lines to support place-value based strategies for addition and subtraction of 2-digit numbers. • Introduce and/or consolidate 3-digit place-value (see Booker et al. for further details), in particular make.

3.2 Additive Strategies Tool

This task assumes some facility with addition and subtraction facts to 20. It examines the extent to which students have access to efficient mental strategies to add and subtract 1 and 2 digit numbers to 30 and beyond, which is an important pre-requisite for developing efficient mental strategies for the multiplication facts to 100.

Observed Response	Interpretation/Suggested Teaching Response
Little/no response to Card 1, counts to 9 by ones or uses fingers.	*May not understand task and/or 'trust the count' for single-digit numbers shown, may not be able to keep track of the count.* • Use subitisation cards to check part-part-whole understanding for numbers to 5 and the extent to which students can recognise and work with numerals to 5 without having to model or count by ones (trusting the count). • Build number fact knowledge (and trusting the count) to 10 using subitisation cards and ten-frames (i.e. recognise 7 is 5 and 2 more, 3 and 4, 1 more than 6, and so on). • Use two large dot dice and/or ten-frame cards to model counting on (i.e. cover number that is known and count on by ones), extend to counting on by 1, 2 or 3 mentally. • Use 2-row bead frames and ten frames to build doubles knowledge to 20.
Experiences difficulty with Card 2 (e.g. takes a long time, uses fingers, taps, nods), and/or indicates that they counted by ones.	*Suggests little or no access to mental strategies beyond count on by 1, 2, or 3 from larger.* • If Students can subitise numbers up to 5, continue building part-part-whole knowledge of numbers to 10 as above, using subitisation cards and ten-frames e.g. 17 is 1 ten and 7 ones. • Develop more efficient addition strategies for number facts to 20: count-on, count-on-from-larger, doubles and near doubles (e.g. for 8 and 9, double 8 is 16 and one more is 17), make-to-ten (e.g. for 6 and 8, simultaneously recognise 8 is 2 less than 10 and 6 is 2 and 4, so 8. 10. 14). • Use "*thinking strings*" to model addition of three or more digits (e.g. for 8 and 5 and 7, record: "*8, 10, 13, 20*").

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3.2 Additive Strategies Tool

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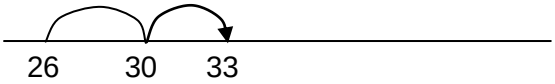
Observed Response	Interpretation/Suggested Teaching Response
Able to find the sum (23) for Card 2 and the missing number (12) for Card 3 but experiences difficulty with Card 4.	*Suggests a knowledge of addition facts to 20 and/or access to relatively efficient mental strategies for adding and subtracting single digit numbers.* • Extend doubling/near doubling strategies to 2-digit numbers emphasising the count of tens (e.g. for double 24, think: double 2 tens is 4 tens , double 4 ones is 8 ones, so 48). • Use open number lines and thinking strings to extend the 'make-to-ten' strategy, eg for 26 and 7  and counting in place-value parts, e.g. for 36 and 47, start at 47, count on 3 tens, 87, and 6 more, 87, 90, 93. • Use similar cards to model and develop "*inspection*" strategies for adding and subtracting 3 two-digit numbers (e.g. for Card 4, 3 tens and 7 tens is 10 tens and 5 more tens is 15 tens, 8 and 2 gives 1 more en 16 tens and 4 more, 164). • Introduce/consolidate column addition for 2 or more addends and trading strategies to support written recording for subtraction (see Booker et al, 2003). • Consider introducing array-based strategies for multiplication facts, e.g. for 2s facts (2 ones, 2 twos, 2 threes, …) think doubles, for 3s facts (3 ones, 3 twos, 3 threes, …) think double the group and one more group etc (See *Developing the Big Ideas in Number* paper in Additional Resources).
Completes all cards reasonably efficiently.	*Indicates sound knowledge of addition and subtraction facts and access to flexible mental strategies for adding and subtracting 2-digit numbers.* • Consolidate written recording for addition and subtraction with regrouping and trading (See Booker et al, 2003). • Extend multiplication strategies to 100 and beyond (e.g. 9 twenty-threes, think: less than 10 twenty-threes (230), one group less, 207).

3.3 Sharing Tool

This task examines the extent to which students understand sharing and the importance of fair shares (equal groups) as a basis for division. It also explores the extent to which students recognise the invariance of the product (in this case that 3 groups of 4 is the same as 4 groups of 3) and the idea of "*times as many*".

Observed Response	Interpretation/Suggested Teaching Response
Little/no response or incomplete (e.g. puts one or two counters on each bag but leaves the rest), hesitant/slow to respond to Card task.	*May not understand task.* • Review idea of sharing and fair shares using appropriate real-world tasks (e.g. share out the glue-pots, so that each table has 3 pots). • Play simple card games that involve dealing out an equal number of cards to each player.
Distributes counters to bags unequally (e.g. makes groups of 6 leaving 2 bags empty or randomly allocates counters to bags). Says there are a different number of dots on each card (either more on Card A because there are more groups or more on Card B because there is more in each group).	*Suggests that task of sharing not understood and/or an inability to discriminate between the number of groups and the number in each group, possibly relying more on perception than reasoning.* • Review idea of sharing and fair shares using appropriate real-world activities (e.g. sharing resources, playing card games). • Make and name arrays, rotate and use strategies to demonstrate equivalence (e.g. 3 rows of 5 is double 5, and 5 more, 15 … 5 rows of 3 is double, double 3 (12) and 3 more, 15). • Play 'Multiplication Toss' (see Additional Resources) to practice making and naming regions and recognising the commutativity of multiplication.
Arrives at correct solution (4 counters/bag) by trial and error. For Card task may say that there are a different number of dots initially but either counts or uses known facts to correct response. Adds (2+2+2) to find number of lollies.	*May not understood value of systematically sharing to ensure equal outcome, may be uncertain about equivalence of different representations.* • Review systematic sharing strategies and importance of fair-shares (e.g. explore collections such as 24, find and name the different ways in which it can be shared equally, i.e. 2 twelves, 3 eights, 4 sixes, 6 fours, 8 threes, 12 twos, discuss how we can be sure that we have found them all).

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3.3 Sharing Tool

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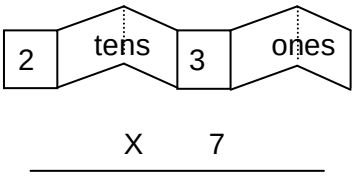
Observed Response	Interpretation/Suggested Teaching Response
Arrives at correct solution (4 counters/bag) by trial and error. For Card task may say that there are a different number of dots initially but either counts or uses known facts to correct response. Adds (2+2+2) to find number of lollies.	• Introduce array-based strategies for multiplication facts, e.g. for 2s facts (2 ones, 2 twos, 2 threes, …) think doubles, for 3s facts (3 ones, 3 twos, 3 threes, …) think double the group and one more group etc (see above). • Play *Multo* (Maths300, Curriculum Corporation) to reinforce the idea that products may be represented in different ways (e.g. 18 is 6 threes, 3 sixes, 9 twos, or 2 nines).
Shares systematically or uses known fact (24 divided by 6 is 4), to arrive at 4/bag. Recognises cards have the same number of dots and determines the number of lollies multiplicatively (3 twos).	*Suggests a sound understanding of initial ideas for multiplication.* • Consolidate mental strategies for multiplication to 100 and beyond. Example: 9 twenty-threes, think: less than 10 twenty-threes (230), one group less, 207. • Introduce symbolic recording for basic facts and discuss informal strategies for multiplying larger numbers by single digit numbers. • Consider introducing the *area* idea for multiplication by using MAB (Base 10 blocks) to explore 1 digit by 2 digit multiplication. Example: Represent 7 by 23 as 7 rows of 2 tens and 3 ones Discuss convenience of this as opposed to 7 rows of 23 ones. • Use Number Expanders to support recording and working with place-value parts (e.g. 7 by 3 ones … 7 by 2 tens). 

3.4 Array and Regions Tool

This task examines the extent to which students are able to work with arrays and regions as models of multiplicative situations. In particular, it explores the strategies used by students to determine the total number of partially hidden items. This provides an important indicator of the student's capacity to work with composite units and his/her readiness to work with more efficient mental strategies for the multiplication facts to 100.

Observed Response	Interpretation/Suggested Teaching Response
Little/no response to Card task or incorrect (e.g. may observe rows and columns or "*grid*", but counts only what can be seen).	*May not understand task and/or "trust the count" for 4 or 6, may not be able to keep track of the count.* • Use subitisation cards to check part-part-whole understanding for numbers to 5 and the extent to which students can recognise and work with numerals to 5 without having to model or count by ones (trusting the count). • Build number fact knowledge (and trusting the count) to 10 using subitisation cards and ten-frames (i.e. recognise 7 is 5 and 2 more, 3 and 4, 1 more than 6, and so on). • Use two large dot dice and/or ten-frame cards to model counting on (i.e. cover number that is known and count on by ones), extend to counting on by 1, 2 or 3 mentally. • Use 2-row bead frames and ten frames to build doubles knowledge to 20.
Counts by ones to get 24, may point to keep track of the hidden dots and/or use partial skip count (e.g. counts by 3s to 18 then counts on top row by ones), experiences difficulty with A3 task (e.g. may not realise cards need to be placed in same orientation).	*May not be able to work with 4 or 6 as a countable unit, may not "trust the count" or be able to keep track of a skip count.* • Use subitisation cards to check part-part-whole understanding for numbers to 10 and the extent to which students 'trust the count' for numbers 5 to 10. • Use ten-frames and 2-row bead frames to build knowledge of doubles, rehearse the use of doubling to more efficiently determine the number in an array (e.g. for 5 rows of 6, think: double 6 is 12, double that again is 24 and 1 more 6 is 30).

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3.4 Array and Regions Tool

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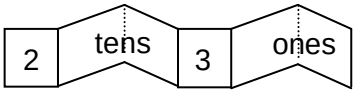
Observed Response	Interpretation/Suggested Teaching Response
More substantive attempt to skip count for initial task (e.g. doubles 6 then doubles again to get 24) and some indication that "*name-tags*" are being used to estimate number of rows and/or columns.	*Able to work with 4 and/or 6 as composite unit, some access to skip counting or more efficient counting strategies, recognises the value of array/region representations but may not be able to make use of these without modelling.* • Consider introducing/consolidating array-based strategies for multiplication facts, e.g. for 2s facts (2 ones, 2 twos, 2 threes, …) think doubles, for 3s facts (3 ones, 3 twos, 3 threes, …) think double the group and one more group etc (See *Developing the Big Ideas in Number* paper in Additional Resources). • Extend to region models and develop multiplication strategies to 100 and beyond (e.g. 9 twenty-threes, think: less than 10 twenty-threes (230), one group less, 207).
Uses known fact to determine 24 dots, systematically arranges "*name-tags*" to determine 20 (if longer edge of cards aligned with longer edge of A3) or 21 (if shorter edge of cards aligned with longer edge of A3).	*Indicates knowledge of relevant number facts and an understanding of the applicability of the region model of multiplication.* • Consolidate mental strategies for multiplication to 100 and beyond (e.g. 9 twenty-threes, think: less than 10 twenty-threes (230), one group less, 207). • Introduce symbolic recording for basic facts and discuss informal strategies for multiplying larger numbers by single digit numbers. • Consider introducing the *area* idea for multiplication by using MAB (Base 10 blocks) to explore 1 digit by 2 digit multiplication (e.g. represent 7 by 23 as 7 rows of 2 tens and 3 ones), discuss convenience of this as opposed to 7 rows of 23 ones. • Use Number Expanders to support recording and working with place-value parts (e.g. 7 by 3 ones … 7 by 2 tens).  X 7

3.5 Cartesian Product Tool

This task examines the extent to which students are able to work with the *Cartesian product* or "*for each*" idea of multiplication. In particular, it explores the strategies used by students to determine the total number of options or combinations. This idea arises in many Chance and Data problems and is needed to support later work with fractions (e.g. fraction renaming) and rate (e.g. 60 kilometres/hour or 3.4 kg @ $1.29/kg) for which the *groups of* idea is no longer applicable.

Observed Response	Interpretation/Suggested Teaching Response
Incorrect (e.g. may add) or no response to T-Shirt task.	*May not understand or be able to represent the task.* • Introduce simple problems like the T-Shirt task (i.e. two variables), discuss how this might be represented and the "for each" idea (e.g. for each colour there are 4 sizes, or for each size there are 3 colours). • Consider introducing tree diagrams as a more efficient way to represent problems of this type.
Solves T-Shirt task (by listing all or multiplying) but irrelevant, incorrect or no response to initial Canteen task.	*Suggests some capacity to model situations involving two variables but may not understand or be able to represent situations involving more than 2 variables.* • Review problems involving two variables (see above) and problems like the initial Canteen task where numbers needed are not stated. • Discuss different ways to represent problems of this type (e.g. tree diagrams, tables) and extend to problems involving three or more variables. • Where students are reliant on representing/counting all, make links to multiplication more explicit .
Solves T-Shirt task and initial Canteen Task (6 options) by listing/counting all options (additive) or by multiplying 2 by 3 (multiplicative), may respond to last question but unable to justify thinking/reasoning.	*Suggests some capacity to model/work with Cartesian product situations but unable to explain or justify their thinking.* • Discuss different ways to represent problems of this type and make links to multiplication more explicit (see above). • Encourage students to construct similar problems and to justify and defend their reasoning to others.
Solves all tasks multiplicatively (i.e. uses multiplication directly without the need to model).	*Indicates relatively sound understanding of Cartesian product idea.* • Provide more complex problems (e.g. from Chance and Data strand) and/or invite students to construct similar problems. • Consider applying the 'for each' idea to fraction models and diagrams (see *Partitioning* paper in the Additional Resources).

3.6 Proportional Reasoning Tool

This classic task examines the extent to which students are able to work with proportional reasoning. In particular, it explores the extent to which students can use "*if … then*" reasoning and multiplicative thinking to solve simple problems of the form a/b = c/d where three values are known.

Observed Response	Interpretation/Suggested Teaching Response
Irrelevant or incorrect response to initial, red eel question.	*May not understand task or appreciate the relationship between the length of the eels and the amount of food.* • Introduce simple proportional reasoning problems involving doubling and/or tripling, model, and discuss possible solution strategies. • Use an extended range of problems to make the links to multiplication and division explicit.
Able to solve the red eel question, and partially solves blue eel question (e.g. halves to get 7 pellets for red eel but unable to say how many for orange eel).	*Suggests that task is understood but solution strategies limited to doubling and halving.* • Use an extended range of proportional reasoning problems that involve more than doubling and halving Example: If 5 drops of nectar feed 2 butterflies, how many drops will be needed to feed 12 butterflies? How many butterflies can be feed by 25 drops? Discuss solution strategies and make the links to multiplication and division explicit. • Consider introducing problems involving non-integer ratios Example: Where red eel is 10 cm long, blue eel is 15 cm long and orange eel is 25 cm long.
Solves all parts of the task multiplicatively (i.e. uses division and multiplication consistently).	*Indicates a relatively sound understanding of simple proportional reasoning.* • Consider introducing problems involving non-integral ratios. Example: Where red eel is 10 cm long, blue eel is 15 cm long and orange eel is 25 cm long.